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Special Article – Tools for Experiment and Theory

Index summation in real time statistical field theory

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Abstract. We have written a *Mathematica* program that calculates the integrand corresponding to any amplitude in the closed-time-path formulation of real time statistical field theory. The program is designed so that it can be used by someone with no previous experience with *Mathematica*. It performs the contractions over the tensor indices that appear in real time statistical field theory and gives the result in the 1-2, Keldysh or RA basis. The program treats all fields as scalars, but the result can be applied to theories with dirac and lorentz structure by making simple adjustments. As an example, we have used the program to calculate the ward identity for the QED 3-point function, the QED 4-point function for two photons and two fermions, and the QED 5-point function for three photons and two fermions. In real time statistical field theory, there are seven 3-point functions, 15 4-point functions and 31 5-point functions. We produce a table that gives the results for all of these functions. In addition, we give a simple general expression for the KMS conditions between n-point green functions and vertex functions, in both the Keldysh and RA bases.

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1 Introduction

It is well known that calculations in real time statistical field theory are complicated by the proliferation of indices that results from the doubling of degrees of freedom (for a recent review see [1]). This difficulty causes many people to avoid using real time finite temperature field theory, in spite of its significant advantages over the imaginary time formalism. Two of the major advantages of working in real time are that analytic continuations are not necessary, and that it is easy to generalize to non-equilibrium situations. In this paper we make a contribution towards reducing the technical difficulties associated with the real time formulation of statistical field theory.

We use the closed time path (CTP) formalism of real time statistical field theory [2, 3] which consists of a contour with two branches: one runs from minus infinity to infinity along the real axis, the other runs back from infinity to minus infinity just below the real axis. This contour results in a doubling of degrees of freedom. Physically, these extra contributions come from the additional processes that are present when the system interacts with a medium, instead of sitting in a vacuum. As a result of these extra degrees of freedom, n-point functions have a tensor stucture which results in calculational complexities that increase geometrically when one considers either calculations at higher loop order, or the calculation of higher n-point functions. Statistical field theory can be formulated in different bases, which produce different representations of these tensors. There are three popular bases: the 1-2 basis, the Keldysh basis, and the RA basis. Most of the original work in this field was done in the 1-2 basis. The Keldysh basis has the advantage of being more easily adaptible to non-equilibrium situations. The RA basis produces particularly simple expressions in equilibrium.

This paper is organized as follows: In Sect. 2 we review the 1-2 formalism. In Sect. 3 we discuss the issue of basis transformations in general. Our discussion follows that of [4]. In Sects. 5 and 6 we give expressions for n-point functions and vertex functions in the Keldysh and RA bases. In Sect. 7 we give the KMS relations in each of these bases. These equations give a set of relations between the various components of a given n-point function or vertex function which hold in equilibrium, and are often useful for simplifying the expressions that result after contracting over indices. We note that some of these expressions have appeared previously in the literature, using a slightly different notation [5–7]. We give general expressions in the Keldysh and RA bases. We emphasize the simplicity of the expressions in the RA basis. In Sect. 8 we discuss the *Mathematica* computer program that we use to perform most of the calculations in this paper. This program is de-

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signed so that it can be used by someone with no previous experience with *Mathematica*. It is available on the internet at www.brandonu.ca/physics/fugleberg/Research/Dick.html.

The program performs contractions over CTP indices and produces the integrand that corresponds to any n-point vertex diagram, in the 1-2, Keldysh or RA basis. Several options to simplify the results are built into the program and can be selected by the user as part of the input. For example, for calculations in equilibrium, the KMS conditions can be automatically implemented. We discuss the basic design of the program and explain in detail, using a simple example, how the program can be used. The program treats all fields as scalar bosons and from the beginning of the paper to the end of Sect. 8 we discuss only scalar bosons. It is straightforward to apply the results of the program to other field theories. In Sect. 9 we illustrate the use of the program with the calculation of some QED ward identities. The general form is the same as the well known zero temperature expression. In real time statistical field theory however, the situation is complicated by the additional degrees of freedom. Each n-point function has 2^n-1 independent components and each of these components has its own ward identity. We start by looking at bare 1-loop diagrams and obtaining the form of the ward identities for the 3-point function ($2^n-1=7$ components), the 4-point function for two fermions and two photons ($2^n-1=15$ components), and the 5-point function for two fermions and three photons ($2^n-1=31$ components). In this paper we produce a complete set of expressions for all 3-, 4- and 5-point functions. For the 3- and 4-point functions, we verify that the same expressions hold when full corrected vertices and propagators are used. Results for some components of the 3-point function were obtained in [8] and [9]. Expressions for bare 1-loop 3- and 4-point diagrams were derived in [7] and [10,11]. For 3- and 4-point functions that have been previously studied, our results agree with those of previous authors, which provides a check of our program. We note that the full calculation involving corrected vertices would be prohibitively difficult without the use of a program like the one presented here. Our conclusions are presented in Sect. 10.

2 The 1-2 basis

The n-point function is defined:

$$G^{(n)}(x_1,\cdots x_n)_{b_1\cdots b_n} := (-i)^{n-1}\langle \mathcal{P}\left(\phi(x_1)_{b_1}\cdots\phi(x_n)_{b_n}\right)\rangle \tag{1}$$

where the subscripts $\{b_i\}$ take values 1 or 2 to indicate which branch of the contour the corresponding time argument falls on, and the symbol $\mathcal{P}$ represents ordering along the closed time path. In what follows we will suppress the superscript (n) and let the number of indices indicate the number of fields in the n-point function. All n-point functions will be denoted G, except for the 2-point function, or the propagator, which will be called D.

In the 1-2 basis the 2-point function can be written as a 2×2 matrix:

$$\mathbf{D}_{(1-2)} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}. \tag{2}$$

The component D_{11} indicates a propagator for fields moving along the top branch of the contour, D_{12} is the propagator for fields moving from the top branch to the bottom branch, etc. In co-ordinate space these components are given by,

$$\begin{aligned} D_{11}(x,y) &= -i\langle T(\phi(x)\phi(y))\rangle \\ D_{12}(x,y) &= -i\langle \phi(y)\phi(x)\rangle \\ D_{21}(x,y) &= -i\langle \phi(x)\phi(y)\rangle \\ D_{22}(x,y) &= -i\langle \tilde{T}(\phi(x)\phi(y))\rangle\,, \end{aligned} \tag{3}$$

where $\mathcal{T}$ is the usual time ordering operator and $\widetilde{\mathcal{T}}$ is the antichronological time ordering operator. These four components satisfy,

$$\sum_{a=1}^{2}\sum_{b=1}^{2}(-1)^{a+b}D_{ab} = D_{11}-D_{12}-D_{21}+D_{22} = 0 \tag{4}$$

as a consequence of the identity $\theta(x)+\theta(-x)=1$, and thus only three components are independent.

The 3-point function in the 1-2 basis is a $(2\times 2\times 2)$ tensor with components in co-ordinate space:

$$\begin{aligned} G_{111}(x,y,z) &= -\langle \mathcal{T}(\phi(x)\phi(y)\phi(z))\rangle \\ G_{112}(x,y,z) &= -\langle \phi(z)\,\mathcal{T}(\phi(x)\phi(y))\rangle \\ G_{121}(x,y,z) &= -\langle \phi(y)\,\mathcal{T}(\phi(x)\phi(z))\rangle \\ G_{211}(x,y,z) &= -\langle \phi(x)\,\mathcal{T}(\phi(y)\phi(z))\rangle \\ G_{122}(x,y,z) &= -\langle \tilde{\mathcal{T}}(\phi(y)\phi(z))\,\phi(x)\rangle \\ G_{212}(x,y,z) &= -\langle \tilde{\mathcal{T}}(\phi(x)\phi(z))\,\phi(y)\rangle \\ G_{221}(x,y,z) &= -\langle \tilde{\mathcal{T}}(\phi(x)\phi(y))\,\phi(z)\rangle \\ G_{222}(x,y,z) &= -\langle \tilde{\mathcal{T}}(\phi(x)\phi(y)\phi(z))\rangle\,. \end{aligned} \tag{5}$$

We note that bosonic green functions are completely symmetric if one permutes simultaneously the CTP index and the co-ordinate. Because of this general property, the equations given above for $G_{121}(x,y,z)$ and $G_{211}(x,y,z)$ can be obtained from the equation for $G_{112}(x,y,z)$ by straightforward permutation. Similarly, the equations for $G_{212}(x,y,z)$ and $G_{221}(x,y,z)$ can be obtained from the equation for $G_{122}(x,y,z)$. In the future, for any set of equations of this type, we will write only one equation and indicate the number of additional permutations. Only seven of these components in (5) are independent because of the identity

$$\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}(-1)^{a+b+c+1}G_{abc} = 0 \tag{6}$$

which follows in the same way as (4) from $\theta(x)+\theta(-x)=1$.

The 4-point function in the 1-2 basis is a $(2\times2\times2\times2)$ tensor. We write out a few examples:

$$\begin{aligned}
G_{1111}(x,y,z,w) &= i\langle\mathcal{T}\phi(x)\phi(y)\phi(z)\phi(w)\rangle\\
G_{1112}(x,y,z,w) &= i\langle\phi(w)\mathcal{T}(\phi(x)\phi(y)\phi(z))\rangle\\
G_{1121}(x,y,z,w) &= i\langle\phi(z)\mathcal{T}(\phi(x)\phi(y)\phi(w))\rangle\\
G_{1211}(x,y,z,w) &= i\langle\phi(y)\mathcal{T}(\phi(x)\phi(z)\phi(w))\rangle\\
G_{2111}(x,y,z,w) &= i\langle\phi(x)\mathcal{T}(\phi(y)\phi(z)\phi(w))\rangle\\
G_{1122}(x,y,z,w) &= i\langle\widetilde{\mathcal{T}}(\phi(z)\phi(w))\mathcal{T}(\phi(x)\phi(y))\rangle\\
&\vdots
\end{aligned} \tag{7}$$

These functions obey a relation similar to (4) and (6):

$$\sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2}(-1)^{a+b+c+d}G_{abcd}=0\,. \tag{8}$$

The structure of higher n-point functions is similar.

Truncated green functions are called vertices and will be denoted Γ, except for the two point function which will be called Π. They are defined by the equation:

$$G_{b_1\cdots b_n}=G_{b_1\bar{b}_1}\cdots G_{b_n\bar{b}_n}\Gamma^{\bar{b}_1\cdots\bar{b}_n}\,. \tag{9}$$

The vertex functions satisfy the constraint:

$$\sum_{b_1=1}^{2}\sum_{b_2=1}^{2}\cdots\sum_{b_n=1}^{2}\Gamma^{b_1\,b_2\cdots b_n}=0\,. \tag{10}$$

3 Basis transformations

A scattering amplitude is calculated by multiplying together a series of vertices and propagators. By convention we will assign lower indices to n-point functions (like the propagator) and upper indices to vertices. Summations are carried out over pairs of repeated indices, where one of the indices is an upper index and the other is a lower index.

The expression for a given scattering amplitude can be transformed to a different basis by performing a rotation. There are two commonly used bases: the Keldysh basis and the R/A basis. Both of these bases express results in terms of combinations of the components of the 1-2 propagator that have a direct physical interpretation: the retarded, advanced and symmetric propagators. These expressions are easy to obtain in co-ordinate space. Using (3) and (4) one can show:

$$\begin{aligned}
D_{\rm ret}(x,y) &= D_{11}(x,y)-D_{12}(x,y)\\
&= -\mathrm{i}\theta(x_0-y_0)\langle[\phi(x),\phi(y)]\rangle\\
D_{\rm adv}(x,y) &= D_{11}(x,y)-D_{21}(x,y)\\
&= -\mathrm{i}\theta(y_0-x_0)\langle[\phi(x),\phi(y)]\rangle\\
D_{\rm sym}(x,y) &= D_{11}(x,y)+D_{22}(x,y)\\
&= -\mathrm{i}\langle\{\phi(x),\phi(y)\}\rangle\,.
\end{aligned} \tag{11}$$

One advantage of the Keldysh basis is that it is easily generalizable to non-equilibrium situations. In equilibrium, the R/A basis produces particularly simple expressions for amplitudes, and for expressions like the KMS conditions, which give relationships between different amplitudes.

The rotation to a different basis is accomplished by matrix multiplication. We rotate lower indices by multiplying by a matrix U and upper indices by multiplying by the matrix V. These matrices are related through the following equation.

$$V(k)=(U^{\rm T})^{-1}(-k)\,. \tag{12}$$

We obtain expressions of the form:

$$\begin{aligned}
&U_{\bar{a}}{}^{a}(k)U_{\bar{b}}{}^{b}(-k)D_{ab}(k)\to D'_{\bar{a},\bar{b}}(k)\\
&V_{a}^{\bar{a}}(k_1)V_{b}^{\bar{b}}(k_2)V_{c}^{\bar{c}}(-k_1-k_2)\Gamma^{abc}(k_1,k_2,-k_1-k_2)\\
&\quad\to(\Gamma')^{\bar{a},\bar{b},\bar{c}}(k_1,k_2,-k_1-k_2)\,,
\end{aligned} \tag{13}$$

where we have simplified the notation by writing a two point function of the form $D_{ab}(k,-k)$ as $D_{ab}(k)$. It is straightforward to see that amplitudes have the correct transformation properties. We look at the example shown in Fig. 1. The amplitude is represented by an expression of the form:

$$\begin{aligned}
\Pi^{ab}(k_1)\sim\int\mathrm{d}k_2\;&\Gamma^{ace}(k_1,k_2,-k_1-k_2)D_{cd}(-k_2)\\
&\times\Gamma^{bdf}(-k_1,-k_2,k_1+k_2)D_{ef}(k_1+k_2)\,.
\end{aligned} \tag{14}$$

Using (12) and (13) it is easy to see that this self-energy transforms according to:

$$(\Pi')^{ab}(k_1)=V^{a}_{\bar{a}}(k_1)V^{b}_{\bar{b}}(-k_1)\Pi^{\bar{a}\bar{b}}(k_1)\,. \tag{15}$$

In the sections below, indices in the 1-2 basis will be denoted b_i and take the values 1 or 2. Keldysh indices will be written α_i and are assigned the values $\alpha=1:=r$ and $\alpha=2:=a$. R/A indices will be denoted X_i and assigned the values $X=1:=R$ and $X=2:=A$. An n-point function in the 1-2 basis will be written:

$$G(k_1,k_2,k_3,\cdots k_n)_{b_1b_2b_3\cdots b_n}:=G_{b_1b_2b_3\cdots b_n}\,. \tag{16}$$

An n-point function in the Keldysh basis will be written:

$$G(k_1,k_2,k_3,\cdots k_n)_{\alpha_1\alpha_2\alpha_3\cdots\alpha_n}:=G_{\alpha_1\alpha_2\alpha_3\cdots\alpha_n}\,. \tag{17}$$

An n-point function in the R/A basis will be written:

$$G(k_1,k_2,k_3,\cdots k_n)_{X_1X_2X_3\cdots X_n}:=G_{X_1X_2X_3\cdots X_n}\,. \tag{18}$$

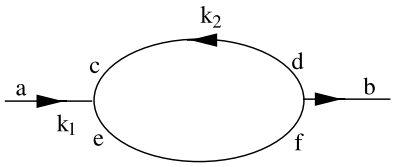

Fig. 1. A typical ampitude

Note that the conservation of momentum gives $k_n = -(k_1 + k_2 + \cdots + k_{n-1})$ so that each n-point function depends on only $n-1$ independent momenta.

4 Thermal functions

We write the general distribution function as $f(k)$ and define the symmetric and anti-symmetric combinations:

$$\mathcal{F}_{\rm A}(k) = f(k) - f(-k)\,, \quad \mathcal{F}_{\rm S}(k) = f(k) + f(-k)\,. \tag{19}$$

In the Keldysh representation it will be useful to define the functions:

$$\begin{aligned} \widetilde{\mathcal{F}}(k_i,\alpha_i) &= \delta_{\alpha_i,a}\mathcal{F}_{\rm A}(k_i) - \delta_{\alpha_i,r}\mathcal{F}_{\rm S}(k_i)\,, \\ \hat{\mathcal{F}}(k_i,\alpha_i) &= \delta_{\alpha_i,r}\mathcal{F}_{\rm A}(k_i) - \delta_{\alpha_i,a}\mathcal{F}_{\rm S}(k_i)\,. \end{aligned} \tag{20}$$

To illustrate these definitions consider the expression:

$$\widetilde{\mathcal{F}}(k_1,\alpha_1)\widetilde{\mathcal{F}}(k_2,\alpha_2)\widetilde{\mathcal{F}}(k_3,\alpha_3)G(k_1,k_2,k_3)_{\alpha_1,\alpha_2,\alpha_3}\,. \tag{21}$$

If we take the case $\alpha_1 = \alpha_2 = r\,,\ \alpha_3 = a$ we obtain:

$$\mathcal{F}_{\rm S}(k_1)\mathcal{F}_{\rm S}(k_2)\mathcal{F}_{\rm A}(k_3)G(k_1,k_2,k_3)_{rra}\,. \tag{22}$$

For bosons in equilibrium these expressions become:

$$\begin{aligned} f(k) &\to n(k) = 1/(\mathrm{e}^{\beta k_0} - 1) \\ \widetilde{\mathcal{F}}(k_i,\alpha_i) &\to \widetilde{N}(k_i,\alpha_i)\,, \quad \hat{\mathcal{F}}(k_i,\alpha_i) \to \hat{N}(k_i,\alpha_i) \\ \mathcal{F}_{\rm S}(k_i) &\to -1\,, \quad \mathcal{F}_{\rm A}(k_i) \to N(k_i)\,, \end{aligned} \tag{23}$$

where we have defined:

$$\begin{aligned} \widetilde{N}(k_i,\alpha_i) &= \delta_{\alpha_i,a}N(k_i) + \delta_{\alpha_i,r}\,, \\ \hat{N}(k_i,\alpha_i) &= \delta_{\alpha_i,r}N(k_i) + \delta_{\alpha_i,a}\,, \\ N_i := N(k_i) &= 1 + 2n(k_i)\,. \end{aligned} \tag{24}$$

Equilibrium functions satisfy:

$$\begin{aligned} &N_1 + N_2 = 0 \quad \text{if} \quad k_2 = -k_1 \\ &1 + N_1N_2 + N_1N_3 + N_2N_3 = 0 \quad \text{if} \quad k_3 = -k_1 - k_2 \\ &N_1 + N_2 + N_3 + N_4 + N_1N_2N_3 + N_2N_3N_4 + N_3N_4N_1 \\ &\quad + N_4N_3N_2 = 0 \quad \text{if} \quad k_4 = -k_1 - k_2 - k_3 \\ &\ \vdots \end{aligned} \tag{25}$$

where the dots indicate that higher order expressions can be generated by iteration. (Expressions with the same properties can be defined for the fermion distribution function: $n_f(k) = 1/(\mathrm{e}^{\beta k_0} + 1)\,, N_F(k_i) = 1 - 2n_f(k_i)$).

In order to write the KMS conditions in a concise way we define the function

$$\begin{aligned} C_n := C(\{N_i\}) &= C(N_1, N_2, \cdots N_n) \\ &= \sum_{p=0}^{n} \frac{1}{2}\Big[1 - (-1)^{n+p}\Big]\mathbf{N}(n,p) \end{aligned} \tag{26}$$

where the symbol $\mathbf{N}(n,p)$ means the following:

[a] start with n indices $\{x_1, x_2, x_3\ \cdots x_n\}$,
[b] consider all possible subsets of these indices containing $p < n$ of the x_i's (without considering order),
[c] for each of these subsets, take the product of the corresponding $N(x_i)$'s,
[d] sum over all sets.

In addition, we define $\mathbf{N}(n,0) = 1$. A few examples will illustrate this notation.

Example [1]: if $n = 3$ and $p = 2$ then the possible sets of p are: $\{x_1, x_2\}$, $\{x_2, x_3\}$, $\{x_3, x_1\}$ and the result is $\mathbf{N}(3,2) = N_1N_2 + N_2N_3 + N_3N_1$.

Example [2]: if $n = 4$ and $p = 3$ then the possible sets of p are: $\{x_1, x_2, x_3\}$, $\{x_2, x_3, x_4\}$, $\{x_3, x_4, x_1\}$, $\{x_4, x_1, x_2\}$ and the result is $\mathbf{N}(4,3) = N_1N_2N_3 + N_2N_3N_4 + N_3N_4N_1 + N_4N_1N_2$.

Example [3]: if $n = 3$ and $p = 1$ then the possible sets of p are: $\{x_1\}$, $\{x_2\}$, $\{x_3\}$ and the result is $\mathbf{N}(3,1) = N_1 + N_2 + N_3$.

Below we write out the first few $C(N_1, N_2, \cdots N_n)$'s:

$$\begin{aligned} C(N_1) &= 1 \\ C(N_1, N_2) &= N_1 + N_2 \\ C(N_1, N_2, N_3) &= 1 + N_1N_2 + N_2N_3 + N_3N_1 \\ C(N_1, N_2, N_3, N_4) &= N_1 + N_2 + N_3 + N_4 + N_1N_2N_3 \\ &\quad + N_2N_3N_4 + N_3N_4N_1 + N_4N_3N_2 \end{aligned} \tag{27}$$

Note that because of (25) each of these expressions is zero if the momenta satisfy $k_1 + k_2 + \cdots k_n = 0$. Using (24) and (25) we have:

$$C(N_1, \cdots N_n) = 2^{n-1}\frac{n(k_1)\cdots n(k_n)}{n\,(k_1 + \cdots + k_n)}\,. \tag{28}$$

To write the KMS conditions for a given n-point function in a compact form, we will need to use C-functions of the form defined above, but with arguments that are not the full set $\{N_i\}$ with i running from 1 to n. We will need the C variables whose arguments are a subset of the N_i's. When working in the Keldysh basis we want C-functions whose arguments are the set of N_i's whose corresponding α_i's take the value [i] r or, [ii] a. When working in the R/A basis we want C-functions whose arguments are the set of N_i's whose the corresponding X_i's take the value [i] R or, [ii] A.

We define these modified C-functions as follows. In the Keldysh basis the set $\{\alpha_i\}$ contains n variables. The number of r's is defined to be n_r and the number of a's is defined to be n_a. Of course, we have $n = n_r + n_a$. We construct a set of n_r variables: $\{N_i\delta_{\alpha_i,r}\}$ and a set of n_a variables: $\{N_i\delta_{\alpha_i,a}\}$. Using these sets of N's we define the corresponding C-functions:

$$\begin{aligned} C_{n_r} &:= C(\{N_i\delta_{\alpha_i,r}\}) \\ C_{n_a} &:= C(\{N_i\delta_{\alpha_i,a}\})\,. \end{aligned} \tag{29}$$

When working in the R/A basis we make the analogous definitions: The set $\{X_i\}$ contains n variables. The number of R's is n_R and the number of A's is n_A. We define:

$$\begin{aligned} C_{n_R} &:= C(\{N_i \delta_{X_i,R}\}) \\ C_{n_A} &:= C(\{N_i \delta_{X_i,A}\}) \,. \end{aligned} \tag{30}$$

To clarify this notation, we look at the following example: Take $n = 7$ and consider the set $\{\alpha_i\} = \{r, r, a, r, a, a, r\}$. We have: $n_r = 4$, $C_{n_r} = C(N_1, N_2, N_4, N_7)$ and $n_a = 3$, $C_{n_a} = C(N_3, N_5, N_6)$.

5 The Keldysh basis

The rotation from the 1-2 representation to the Keldysh representation is accomplished by using the transformation matrix:

$$U_{\text{Keldysh}\leftarrow(1-2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{31}$$

Note that in this case (12) gives $V = U$. In general the n-point function in the Keldysh representation is given by:

$$G_{\alpha_1 \cdots \alpha_n} = 2^{\frac{n}{2}-1} U_{\alpha_1}{}^{b_1} \cdots U_{\alpha_n}{}^{b_n} G_{b_1 \cdots b_n} \,. \tag{32}$$

The factor $2^{n/2-1}$ is introduced to produce simpler expressions for the vertices, following [6] and [12]. For example, using the notation defined in (32), the bare 3-point vertex in the Keldysh representation is written in tensor form:

$$\{\{\{0, -i\}, \{-i, 0\}\}, \{\{-i, 0\}, \{0, -i\}\}\} \,. \tag{33}$$

This expression can be compared with the bare 3-point vertex in the 1-2 representation:

$$\{\{\{-i, 0\}, \{0, 0\}\}, \{\{0, 0\}, \{0, i\}\}\} \,. \tag{34}$$

In addition, the appropriate components of the Keldysh vertex correspond directly to the retarded vertices (see for example (41)).

5.1 2-point function

The 2-point function in the Keldysh basis is obtained from (32):

$$D_{\alpha_1 \alpha_2} = U_{\alpha_1}{}^{b_1} U_{\alpha_2}{}^{b_2} G_{b_1 b_2} \,, \tag{35}$$

and can be represented (using (4) and (11)) by:

$$\begin{aligned} \mathbf{D}_{\text{Keldysh}} :&= \begin{pmatrix} D_{rr} & D_{ra} \\ D_{ar} & D_{aa} \end{pmatrix} \\ &= U \cdot \mathbf{D}_{(1-2)} \cdot U^T = \begin{pmatrix} D_{11} + D_{22} & D_{11} - D_{12} \\ D_{11} - D_{21} & 0 \end{pmatrix} \\ &= \begin{pmatrix} D_{\text{sym}} & D_{\text{ret}} \\ D_{\text{adv}} & 0 \end{pmatrix} . \end{aligned} \tag{36}$$

Note that $D_{aa} = 0$. In fact, for any n-point function in the Keldysh basis, it is always true that

$$G_{aaa\cdots a} = 0 \,. \tag{37}$$

To understand this point, recall that according to (4), (6) and (8) it is always possible to express one component of the tensor 2-, 3- and 4-point functions in terms of the other components. It is straightforward to show that in the 1-2 basis a constraint of the same form as (4), (6) and (8) exists for any arbitrary n-point function. In the Keldysh representation this constraint takes the form (37). The vertex function obeys an analogous constraint:

$$\Gamma^{rrr\cdots r} = 0 \,. \tag{38}$$

5.2 3-point function

The 3-point function in the Keldysh representation is obtained from (32):

$$G_{\alpha_1 \alpha_2 \alpha_3} = \sqrt{2}\, U_{\alpha_1}{}^{b_1} U_{\alpha_2}{}^{b_2} U_{\alpha_3}{}^{b_3} G_{b_1 b_2 b_3} \,. \tag{39}$$

We give three examples of the results that we obtain by summing over indices and using (6):

$$\begin{aligned} G_{rrr} &= G_{111} + G_{221} + G_{212} + G_{122} \\ G_{rra} &= G_{111} - G_{112} + G_{221} - G_{222} \\ G_{raa} &= G_{111} - G_{112} - G_{121} + G_{122} \,. \end{aligned} \tag{40}$$

Using (5) it is easy to show in co-ordinate space that

$$G_{raa}(x_1, x_2, x_3) = G_{R_1}(x_1, x_2, x_3) \,, \tag{41}$$

where G_{R_1} is the component of the 3-point function that is retarded with respect to the first leg and is given by the familiar expression:

$$\begin{aligned} &G_{R_1}(x_1, x_2, x_3) \\ &= (-\mathrm{i})^2 \Big(\theta(t_1 - t_2)\theta(t_2 - t_3) \langle [[\phi(x_1), \phi(x_2)], \phi(x_3)] \rangle \\ &\qquad + \theta(t_1 - t_3)\theta(t_3 - t_2) \langle [[\phi(x_1), \phi(x_3)], \phi(x_2)] \rangle \Big) \\ &= (-\mathrm{i})^2 \sum_{\{t_a, t_b\} = \{t_2, t_3\}} \theta(t_1, t_a)\theta(t_a, t_b) \\ &\qquad \times \langle [[\phi(t_1), \phi(t_a)], \phi(t_b)] \rangle \,. \end{aligned} \tag{42}$$

The summation indicates that we sum over the combinations: $(t_a = t_2,\ t_b = t_3)$ and $(t_a = t_3,\ t_b = t_2)$. The other two components given in (40) correspond to similar expressions in co-ordinate space, but involve anti-commutators.

5.3 4-point function

The 4-point function in the Keldysh representation is obtained from (32):

$$G_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = 2 U^{b_1}_{\alpha_1} U^{b_2}_{\alpha_2} U^{b_3}_{\alpha_3} U^{b_4}_{\alpha_4} G_{b_1 b_2 b_3 b_4} \,. \tag{43}$$

We give one example of the results that we obtain by summing over indices and using (8):

$$G_{raaa} = G_{1111} - G_{1112} - G_{1121} - G_{1211} + G_{1122} + G_{1212} + G_{1221} - G_{1222}\,. \tag{44}$$

In co-ordinate space this expression has the form:

$$G_{raaa}(x_1,x_2,x_3,x_4) = G_{R1}(x_1,x_2,x_3,x_4) = (-i)^3 \sum_{\{t_a,t_b,t_c\}=\{t_2,t_3,t_4\}} \theta(t_1,t_a)\theta(t_a,t_b)\theta(t_b,t_c) \langle[[[\phi(t_1),\phi(t_a)],\phi(t_b)],\phi(t_c)]\rangle\,, \tag{45}$$

where the sum is over all possible assignments of the variables $\{t_2,\, t_3,\, t_4\}$ to the variables $\{t_a,\, t_b,\, t_c\}$ and the notation $G_{R1}(x_1,x_2,x_3,x_4)$ indicates the 4-point function that is retarded with respect to the first leg. Expressions for higher n-point functions are obtained similarly.

6 The (R/A) basis

The matrix that performs rotations from the 1-2 basis to the R/A basis is:

$$U_{(R/A)\leftarrow(1-2)} = \begin{pmatrix} -f(k) & f(k) \\ 1 & f(k)/f(-k) \end{pmatrix}. \tag{46}$$

We will find it more useful to obtain expressions in the RA basis by rotating from the Keldysh basis. The matrix that transforms from the Keldysh basis to the 1-2 basis is given by the inverse of (31). Combining with (46) we obtain the matrix that transforms from the Keldysh basis to the R/A basis:

$$U_{(R/A)\leftarrow\text{Keldysh}} = U_{(R/A)\leftarrow(1-2)}(U_{\text{Keldysh}\leftarrow(1-2)})^{-1} = \begin{pmatrix} 0 & -\sqrt{2}\,f(k) \\ \frac{\mathcal{F}_{\rm S}(-k)}{\sqrt{2}\,f(-k)} & \frac{\mathcal{F}_{\rm A}(-k)}{\sqrt{2}\,f(-k)} \end{pmatrix}. \tag{47}$$

The n-point function in R/A basis is given by:

$$2^{\frac{n}{2}-1} G_{X_1\cdots X_n} = U(k_1)^{\alpha_1}_{X_1}\cdots U(k_n)^{\alpha_n}_{X_n} G_{\alpha_1\cdots\alpha_n}\,. \tag{48}$$

General expressions that relate R/A n-point functions and vertex functions to Keldysh functions are given below:

$$\begin{aligned} &G(k_1,\cdots k_n)_{X_1\cdots X_n} \\ &= \frac{(-1)^n}{2^{n-1}} \frac{\prod_i[\delta_{X_i,A} + \delta_{X_i,R}\,2\,f(k_i)]}{\prod_i[\delta_{X_i,R} + \delta_{X_i,A}\,f(-k_i)]} \\ &\quad\times \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\widetilde{\mathcal{F}}(k_l;\alpha_l)\delta_{X_lA} + \delta_{X_lR}\delta_{\alpha_l a}\right]\right) \\ &\quad\times G(k_1,\cdots k_n)_{\alpha_1\cdots\alpha_n} \end{aligned}$$

$$\begin{aligned} &\Gamma(k_1,\cdots k_n)^{X_1\cdots X_n} \\ &= \frac{(-1)^n}{2^{n-1}} \frac{\prod_i[\delta_{X_i,R} + \delta_{X_i,A}\,2\,f(k_i)]}{\prod_i[\delta_{X_i,A} + \delta_{X_i,R}\,f(-k_i)]\,(-\mathcal{F}_{\rm S}(k_i))} \\ &\quad\times \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\hat{\mathcal{F}}(k_l;\alpha_l)\delta_{X_lR} + \delta_{X_lA}\delta_{\alpha_l r}\right]\right) \\ &\quad\times \Gamma(k_1,\cdots k_n)^{\alpha_1\cdots\alpha_n}\,. \end{aligned} \tag{49}$$

From (37), (38) and (49) we have

$$G_{R\cdots R} = 0\,, \quad \Gamma^{A\cdots A} = 0\,. \tag{50}$$

These equations can be written in a simpler form in equilibrium. There are two kinds of simplifications that occur. We can replace general distribution functions with thermal distribution functions and use identities of the form (23)–(25), (28) for combining groups of distributions functions. We can also use KMS conditions. The KMS conditions are a set of equations which are valid only at equilibrium that relate various components of an n-point function or vertex function. These conditions will be derived in the next section. For completeness, we give below two general expressions: the first corresponds to (49) using thermal distribution functons, and the second comes from (49) using thermal distribution functions and the KMS conditions. The general equation obtained from (49) using thermal distribution functions is:

$$\begin{aligned} &2^{n_R-1}(-1)^{n_R+1}\,C_{n_A}(\{N_i\delta_{X_i,A}\})G(k_1,\cdots k_n)_{X_1\cdots X_n} \\ &= C_{n_R}(\{N_i\delta_{X_i,R}\}) \\ &\quad\times \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\widetilde{N}(k_l;\alpha_l)\delta_{X_lA} + \delta_{X_lR}\delta_{\alpha_l a}\right]\right) \\ &\quad\times G(k_1,\cdots k_n)_{\alpha_1\cdots\alpha_n}. \\ &2^{n_{\rm A}-1}(-1)^{n_A+1}\,C_{n_R}(\{N_i\delta_{X_i,R}\})\Gamma(k_1,\cdots k_n)^{X_1\cdots X_n} \\ &= C_{n_A}(\{N_i\delta_{X_i,A}\}) \\ &\quad\times \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\hat{N}(k_l;\alpha_l)\delta_{X_lR} + \delta_{X_lA}\delta_{\alpha_l r}\right]\right) \\ &\quad\times \Gamma(k_1,\cdots k_n)^{\alpha_1\cdots\alpha_n}\,. \end{aligned} \tag{51}$$

After applying the KMS conditions to the right hand side of this expression we obtain:

$$\begin{aligned} &2^{n_R-1}(-1)^{n_R+1}\,G(k_1,\cdots k_n)_{X_1\cdots X_n} \\ &= \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\widetilde{N}(k_l;\alpha_l)\delta_{X_lR} + \delta_{X_lA}\delta_{\alpha_l a}\right]\right) \\ &\quad\times G^*(k_1,\cdots k_n)_{\alpha_1\cdots\alpha_n}. \end{aligned} \tag{52}$$

For clarity we give a few examples. We first give general expressions using (49). We next give the results obtained with thermal distribution functions using (51). Finally, we give the expressions obtained by applying the KMS conditions to the right hand side of the previous equations, or equivalently from using (52).

[i] The 2-point functions in the R/A formalism are given by:

$$\begin{aligned} D_{RR} &= 0 \\ D_{RA} &= -\mathcal{F}_S(k_2) D_{ar} \\ D_{AR} &= -\mathcal{F}_S(k_1) D_{ra} \\ D_{AA} &= \left(\mathcal{F}_S(k_1)\mathcal{F}_S(k_1) D_{rr} - \mathcal{F}_A(k_1)\mathcal{F}_S(k_2) D_{ar} \right. \\ &\quad \left. - \mathcal{F}_S(k_1)\mathcal{F}_R(k_2) D_{ra}\right) / \left(2 f(-k_1) f(-k_2)\right). \end{aligned} \tag{53}$$

Using equilibrium distribution functions these results reduce to:

$$\begin{aligned} &D_{RR} = 0\,, \quad D_{RA} = D_{ar}\,, \quad D_{AR} = D_{ra}\,, \\ &D_{AA} = \frac{1}{2n(-k_1) n(-k_2)} \left(D_{rr} + N_1 D_{ar} + N_2 D_{ra}\right). \end{aligned}$$

Applying KMS conditions we have:

$$D_{RR} = 0\,, \quad D_{RA} = D^*_{ra}\,, \quad D_{AR} = D^*_{ar}\,, \quad D_{AA} = 0\,,$$

which can be written in the familiar form:

$$\mathbf{D}_{R/A} = \begin{pmatrix} D_{RR} & D_{RA} \\ D_{AR} & D_{AA}\,, \end{pmatrix} = \begin{pmatrix} 0 & D_{\rm adv} \\ D_{\rm ret} & 0 \end{pmatrix}. \tag{54}$$

[ii] The 3-point functions in the R/A formalism are given by:

$$G_{RRA} = \frac{f(k_1) f(k_2)}{f(k_1 + k_2)} \mathcal{F}_S(k_3)\, G_{aar} \quad \text{(and two permutations)} \tag{55}$$

$$\begin{aligned} G_{AAR} = &-\frac{1}{2} \frac{f(-k_1 - k_2)}{f(-k_1) f(-k_2)} \left(\mathcal{F}_S(k_1)\mathcal{F}_S(k_2) G_{rra} \right. \\ &\left. - \mathcal{F}_A(k_1)\mathcal{F}_S(k_2) G_{ara} - \mathcal{F}_S(k_1)\mathcal{F}_A(k_2) G_{raa}\right) \\ &\text{(and two permutations)} \\ G_{AAA} = &\frac{1}{4\, f(-k_1)\, f(-k_2)\, f(-k_3)} \\ &\times \Big(\mathcal{F}_S(k_1)\mathcal{F}_S(k_2)\mathcal{F}_S(k_3) G_{rrr} \\ &- \mathcal{F}_A(k_1)\mathcal{F}_S(k_2)\mathcal{F}_S(k_3) G_{arr} \\ &- \mathcal{F}_S(k_1)\mathcal{F}_A(k_2)\mathcal{F}_S(k_3) G_{rar} \\ &- \mathcal{F}_S(k_1)\mathcal{F}_S(k_2)\mathcal{F}_A(k_3) G_{rra} \\ &+ \mathcal{F}_A(k_1)\mathcal{F}_A(k_2)\mathcal{F}_A(k_3) G_{aar} \\ &+ \mathcal{F}_A(k_1)\mathcal{F}_S(k_2)\mathcal{F}_A(k_3) G_{ara} \\ &+ \mathcal{F}_S(k_1)\mathcal{F}_A(k_2)\mathcal{F}_A(k_3) G_{raa} \Big), \end{aligned}$$

where indices and momenta must be permuted simultaneously as discussed following (5).

Using equilibrium distribution functions these results reduce to:

$$\begin{aligned} G_{RRA} &= -\frac{(N_1 + N_2)}{2}\, G_{aar} \quad \text{(and two permutations)} \\ G_{AAR} &= \frac{1}{(N_1 + N_2)} \left(G_{rra} + N_1 G_{ara} + N_2 G_{raa}\right) \\ &\quad \text{(and two permutations)} \\ G_{AAA} &= -\frac{1}{4\, n(-k_1)\, n(-k_2)\, n(-k_3)} \\ &\quad \times \left(G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} \right. \\ &\quad \left. + N_1 N_2 G_{aar} + N_1 N_3 G_{ara} + N_2 N_3 G_{raa}\right). \end{aligned} \tag{56}$$

Applying KMS conditions we have:

$$\begin{aligned} G_{RRA} &= -\frac{1}{2} \left(G^*_{rra} + N_1 G^*_{ara} + N_2^* G_{raa}\right) \\ G_{RAR} &= -\frac{1}{2} \left(G^*_{rar} + N_1 G^*_{aar} + N_3 G^*_{raa}\right) \\ G_{ARR} &= -\frac{1}{2} \left(G^*_{arr} + N_2 G^*_{aar} + N_3 G^*_{ara}\right) \\ G_{AAR} &= G^*_{aar} \\ G_{ARA} &= G^*_{ara} \\ G_{RAA} &= G^*_{raa} \\ G_{AAA} &= 0\,. \end{aligned} \tag{57}$$

For the 4-point functions using equilibrium distribution functions we have:

$$\begin{aligned} G_{RRRA} &= \frac{C(N_1, N_2, N_3)}{4}\, G_{aaar} \quad \text{(and three permutations)} \\ G_{RRAA} &= -\frac{1}{2} \frac{C(N_1, N_2)}{C(N_3, N_4)} \left(G_{aarr} + N_3 G_{aaar} + N_4 G_{aara}\right) \\ &\quad \text{(and five permutations)} \\ G_{AAAR} &= \frac{1}{C(N_1, N_2, N_3)} \\ &\quad \times \left(G_{rrra} + N_1 G_{arra} + N_2 G_{rara} + N_3 G_{rraa} \right. \\ &\quad \left. + N_1 N_2 G_{aara} + N_1 N_3 G_{araa} + N_2 N_3 G_{raaa}\right) \\ &\quad \text{(and three permutations)} \\ G_{AAAA} &= \frac{1}{8\; n(-k_1)\, n(-k_2)\, n(-k_3)\, n(-k_4)} \\ &\quad \times \Big(G_{rrrr} + N_1 G_{arrr} + N_2 G_{rarr} + N_3 G_{rrar} \\ &\quad + N_4 G_{rrra} + N_1 N_2 G_{aarr} + N_1 N_3 G_{arar} \\ &\quad + N_1 N_4 G_{arra} + N_2 N_3 G_{raar} + N_2 N_4 G_{rara} \\ &\quad + N_3 N_4 G_{rraa} + N_1 N_2 N_3 G_{aaar} \\ &\quad + N_1 N_2 N_4 G_{aara} + N_1 N_3 N_4 G_{araa} \\ &\quad + N_2 N_3 N_4 G_{raaa}\Big). \end{aligned} \tag{58}$$

Applying KMS conditions we have:

$$\begin{aligned} G_{RRRA} &= \frac{1}{4}\big(G^*_{rrra} + N_1 G^*_{arra} + N_2 G^*_{rara} + N_3 G^*_{rraa} \\ &\quad + N_1 N_2 G^*_{aara} + N_1 N_3 G^*_{araa} + N_2 N_3 G^*_{raaa}\big) \\ &\quad \text{(and three permutations)} \\ G_{RRAA} &= -\frac{1}{2} \left(G^*_{rraa} + N_1 G^*_{araa} + N_2 G^*_{raaa}\right) \\ &\quad \text{(and five permutations)} \\ G_{AAAR} &= G^*_{aaar} \quad \text{(and three permutations)} \\ G_{AAAA} &= 0\,. \end{aligned} \tag{59}$$

7 KMS conditions

The KMS conditions are a set of relations between the various components of a given n-point function that hold at equilibrium. The KMS conditions are often useful for simplifying the expressions that result from the contractions

over indices, in any representation of real time statistical field theory. The KMS conditions have the simplest structure when expressed in the R/A basis.

7.1 Keldysh basis

The KMS conditions in the Keldysh representation can be written:

$$\begin{aligned} & C_{n_a}(\{N_i\delta_{\alpha_i',a}\}) \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\widetilde{N}(k_l;\alpha_l)\delta_{\alpha_l' r}+\delta_{\alpha_l' a}\delta_{\alpha_l a}\right]\right) \\ & \quad \times G_{\alpha_1\cdots\alpha_n} \\ =& C_{n_r}(\{N_i\delta_{\alpha_i',r}\}) \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\widetilde{N}(k_l;\alpha_l)\delta_{\alpha_l' a}+\delta_{\alpha_l' r}\delta_{\alpha_l a}\right]\right) \\ & \quad \times G^*_{\alpha_1\cdots\alpha_n} \end{aligned} \tag{60}$$

In this master equation, the set of variables $\{\alpha_i'\}$ are external variables that are not summed over. Thus, in principle, (60) contains 2^n equations which come from the 2^n choices of the sets $\{\alpha_i'\}$. In fact, one half of these equations is the complex conjugate of the other half and thus we have 2^{n-1} KMS conditions for each n-point function. We give several examples below:

$[n=2]$

For the 2-point function we obtain $2^{n-1}\Big|_{n=2} = 2$ equations:

$$\begin{aligned} & D_{ra}(p) = D^*_{ar}(p) \quad \text{or} \quad D_{\rm ret}(p) = D^*_{\rm adv}(p) \\ & D_{rr}(p) = N(p)(D_{ra}(p) - D_{ar}(p)) \quad \text{or} \\ & D_{\rm sym}(p) = N(p)(D_{\rm ret}(p) - D_{\rm adv}(p))\,. \end{aligned} \tag{61}$$

$[n=3]$

There are $2^{n-1}\Big|_{n=3} = 4$ independent KMS conditions which are:

$$\begin{aligned} & G_{rrr} + N_1 G_{arr} + N_2 G_{rar} + N_3 G_{rra} + N_1N_2 G_{aar} \\ & + N_1N_3 G_{ara} + N_2N_3 G_{raa} = 0 \\ & G_{rra} + N_1 G_{ara} + N_2 G_{raa} = (N_1+N_2) G^*_{aar} \\ & \quad \text{(and two permutations)} \end{aligned} \tag{62}$$

$[n=4]$

There are $2^{n-1}\Big|_{n=4} = 8$ independent KMS conditions which are:

$$\begin{aligned} & G_{rrrr} + N_1 G_{arrr} + N_2 G_{rarr} + N_3 G_{rrar} + N_4 G_{rrra} \\ & + N_1N_2 G_{aarr} + N_1N_3 G_{arar} + N_1N_4 G_{arra} \\ & + N_2N_3 G_{raar} + N_2N_4 G_{rara} + N_3N_4 G_{rraa} \\ & + N_1N_2N_3 G_{aaar} + N_1N_2N_4 G_{aara} \\ & + N_1N_3N_4 G_{araa} + N_2N_3N_4 G_{raaa} = 0 \end{aligned} \tag{63}$$

$$\begin{aligned} & G_{rrra} + N_1 G_{arra} + N_2 G_{rara} + N_3 G_{rraa} + N_1N_2 G_{aara} \\ & + N_1N_3 G_{araa} + N_2N_3 G_{raaa} \\ & = (1 + N_1N_2 + N_1N_3 + N_2N_3) G^*_{aaar} \\ & \quad \text{(and three permutations)} \\ & (N_3+N_4)\,(G_{rraa} + N_1 G_{araa} + N_2 G_{raaa}) \\ & = (N_1+N_2)\,(G^*_{aarr} + N_3 G^*_{aaar} + N_4 G^*_{aara}) \\ & \quad \text{(and two permutations)} \end{aligned} \tag{64}$$

$[n=5]$

There are $2^{n-1}\Big|_{n=5} = 16$ independent KMS conditions. We write down three representative ones:

$$\begin{aligned} & G_{rrrrr} + N_1 G_{arrrr} + N_2 G_{rarrr} + N_3 G_{rrarr} + N_4 G_{rrrar} \\ & + N_5 G_{rrrra} + N_1N_2 G_{aarrr} + N_1N_3 G_{ararr} + N_1N_4 G_{arrar} \\ & + N_1N_5 G_{arrra} + N_2N_3 G_{raarr} + N_2N_4 G_{rarar} \\ & + N_2N_5 G_{rarra} + N_3N_4 G_{rraar} + N_3N_5 G_{rrara} \\ & + N_4N_5 G_{rrraa} + N_1N_2N_3 G_{aaarr} + N_1N_2N_4 G_{aarar} \\ & + N_1N_2N_5 G_{aarra} + N_1N_3N_4 G_{araar} + N_1N_3N_5 G_{arara} \\ & + N_1N_4N_5 G_{arraa} + N_2N_3N_4 G_{raaar} + N_2N_3N_5 G_{raara} \\ & + N_2N_4N_5 G_{raraa} + N_3N_4N_5 G_{rraaa} + N_1N_2N_3N_4 G_{aaaar} \\ & + N_1N_2N_3N_5 G_{aaara} + N_1N_2N_4N_5 G_{aaraa} \\ & + N_1N_3N_4N_5 G_{araaa} + N_2N_3N_4N_5 G_{raaaa} = 0 \end{aligned}$$

$$\begin{aligned} & G_{rrrra} + N_1 G_{arrra} + N_2 G_{rarra} + N_3 G_{rrara} + N_4 G_{rrraa} \\ & + N_1N_2 G_{aarra} + N_1N_3 G_{arara} + N_1N_4 G_{arraa} \\ & + N_2N_3 G_{raara} + N_2N_4 G_{raraa} + N_3N_4 G_{rraaa} \\ & + N_1N_2N_3 G_{aaara} + N_1N_2N_4 G_{aaraa} + N_1N_3N_4 G_{araaa} \\ & + N_2N_3N_4 G_{raaaa} \\ & = (N_1 + N_2 + N_3 + N_4 + N_1N_2N_3 + N_1N_2N_4 + N_1N_3N_4 \\ & \quad + N_2N_3N_4) G^*_{aaaar} \end{aligned}$$

$$\begin{aligned} & (N_4+N_5)(G_{rrraa} + N_1 G_{arraa} + N_2 G_{raraa} + N_3 G_{rraaa} \\ & + N_1N_2 G_{aaraa} + N_1N_3 G_{araaa} + N_2N_3 G_{raaaa}) \\ & = (1 + N_1N_2 + N_1N_3 + N_2N_3) \\ & \quad \times (G^*_{aaarr} + N_4 G^*_{aaaar} + N_5 G^*_{aaara}) \end{aligned} \tag{65}$$

The KMS conditions for the vertex functions have almost exactly the same form. The master equation is obtained from (60) by interchanging the indices r and a:

$$\begin{aligned} & C_{n_r}(\{N_i\delta_{\alpha_i',r}\}) \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\hat{N}(k_l;\alpha_l)\delta_{\alpha_l' a}+\delta_{\alpha_l' r}\delta_{\alpha_l r}\right]\right) \\ & \quad \times \Gamma^{\alpha_1\cdots\alpha_n} \\ & = C_{n_a}(\{N_i\delta_{\alpha_i',a}\}) \\ & \quad \times \sum_{\alpha_l\in\{r,a\}} \left(\prod_{l=1}^{n}\left[\hat{N}(k_l;\alpha_l)\delta_{\alpha_l' r}+\delta_{\alpha_l' a}\delta_{\alpha_l r}\right]\right) (\Gamma^*)^{\alpha_1\cdots\alpha_n}\,. \end{aligned} \tag{66}$$

7.2 R/A basis

The KMS conditions have an even simpler form in the R/A representation. This can be anticipated by comparing the general structure of (51) and (60). The general expression for the KMS conditions in the R/A basis

is:

$$2^{n_R-n_A}\, C_{n_A}(\{N_i\delta_{X_i,A}\})\, G_{X_1\cdots X_n} = (-1)^n C_{n_{\bar{A}}}(\{N_i\delta_{X_i,\bar{A}}\})\, G^*_{\bar{X}_1\cdots\bar{X}_n}\,. \quad (67)$$

This master equation contains a series of equations where the variables X_i take all possible combinations of the values $\{R, A\}$. We use the notation: $\bar{A} = R$, $\bar{R} = A$ so that $n_{\bar{R}} = n_A$ and $n_{\bar{A}} = n_R$. We give several examples below:

$[n = 2]$ For the 2-point functions we have:

$$D_{AR}(p) = D^*_{RA}(p)\,, \quad D_{AA} = 0\,. \quad (68)$$

$[n = 3]$ For the 3-point functions we have:

$$\begin{aligned} G_{AAA} &= 0 \\ 2C(N_3)G_{RRA} &= -C(N_1,N_2)G^*_{AAR} \rightarrow \\ 2G_{RRA} &= -(N_1+N_2)G^*_{AAR} \\ &\quad \text{(and two permutations)} \end{aligned} \quad (69)$$

where the arrows indicate the results that are obtained by replacing the C-functions by their definitions in terms of thermal functions (see (26)).

$[n = 4]$ For the 4-point functions we have:

$$\begin{aligned} G_{AAAA} &= 0 \\ C(N_3,N_4)G_{RRAA} &= C(N_1,N_2)G^*_{AARR} \rightarrow \\ (N_3+N_4)G_{RRAA} &= (N_1+N_2)G^*_{AARR} \\ &\quad \text{(and two permutations)} \\ 4G_{ARRR} &= C(N_2,N_3,N_4)G^*_{RAAA} \rightarrow \\ 4G_{ARRR} &= (1+N_2N_3+N_3N_4+N_4N_2)G^*_{RAAA} \\ &\quad \text{(and three permutations)} \end{aligned} \quad (70)$$

$[n = 5]$ For the 5-point functions we give a few examples:

$$\begin{aligned} &G_{AAAAA} = 0 \quad (71)\\ &C(N_3,N_4,N_5)G_{RRAAA} = -2C(N_1,N_2)G^*_{AARRR} \rightarrow \\ &(1+N_3N_4+N_4N_5+N_5N_3)G_{RRAAA} = -2(N_1+N_2) \\ &\quad \times G^*_{AARRR} \\ &8G_{RRRRA} = -C(N_1,N_2,N_3,N_4)G^*_{AAAAR} \rightarrow \\ &8G_{RRRRA} = -(N_1+N_2+N_3+N_4+N_1N_2N_3+N_2N_3N_4 \\ &\quad + N_3N_4N_1+N_4N_1N_2)G^*_{AAAAR} \end{aligned}$$

Note that in every case one of the KMS equations requires the vanishing of the $G_{A\ldots A}$ component of the n-point function. The rest of the KMS equations are simple relations between pairs of off-diagonal components.

The KMS conditions for the vertex functions have almost exactly the same form. They are obtained from the same master equation, with the indices R and A interchanged:

$$2^{n_A-n_R}\, C_{n_R}(\{N_i\delta_{X_i,R}\})\, \Gamma^{X_1\cdots X_n} = (-1)^n C_{n_{\bar{R}}}(\{N_i\delta_{X_i,\bar{R}}\})\, (\Gamma^*)^{\bar{X}_1\cdots\bar{X}_n}\,. \quad (72)$$

8 A program to perform the contraction of indices

8.1 Description of the program

Our program calculates the integrand for any diagram by contracting indices. The calculation can be done in the 1-2, Keldysh or R/A basis. The basic strategy of the program is to treat a Feynman diagram as a tensor product of propagators, and bare and corrected vertices. In principle, this type of calculation can be done by hand, but when working with more than a few indices the process becomes extremely tedious. The symbolic manipulation program *Mathematica* is ideally suited to perform this kind of tensor calculation.

The program is divided into six main sections. Only the *Input* section needs to be edited by the user. The user enters some variables in this section, in order to specify the diagram that he wants to calculate. The rest of the program can be immediately executed, and the result is output in the last section. The user has the option to output the results to a file. The results are also defined functionally. Some basic functions are defined within the program that can be used interactively to manipulate the result.

When working in the RA basis the program always assumes thermal distribution functions, and thus the result is only valid in equilibrium. The user has the option to implement the KMS conditions, since both forms of the result can be useful. In the Keldysh basis, thermal distribution forms are only assumed if the KMS conditions are used. In the 1-2 basis the KMS conditions cannot be implemented in the current version of the program. Note that we have not used different notation to indicate equilibrium and non-equilibrium distributions (see (23) and (24)): the output of the program will always contain distribution functions written in the form n_p or N_P.

The program is designed to contract CTP indices and is written for scalar bosons. We set all coupling constants to one and use the notation:

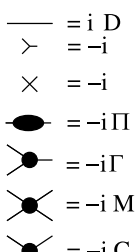

Fig. 2. Definitions of notation for propagator and vertices

Note that additional numerical factors are introduced in the Keldysh representation, as defined in (32). The program can be used for other field theories with appropriate adjustments. The appropriate coupling constant(s) must be inserted by the user. In addition, any dirac, lorentz, or other group structure must be separately handled by the user.

We describe briefly the main sections of the program:

1. The **Initialization** section inputs the necessary *Mathematica* package.
2. The **Input** section is edited by the user to input specific parameters corresponding to the diagram he wants to calculate. This process is described in detail below.
3. The **Definitions** section establishes some basic definitions that will be used throughout the program.
4. The **Find Loops** section identifies all closed loops in the diagram. The variable (*loopzerosubs*) is printed out. The usefulness of this variable is explained in point 11 below.
5. The **Calculate Diagram** section performs the calculation.
6. The **Results** section outputs the results of the calculations and performs some basic manipulations to simplify them.

8.2 Input section

We illustrate the **Input** section of the program with an example. Consider the diagram in Fig. 3.
The **Input** section for this diagram is reproduced in Fig. 4. Each entry is explained in the order it appears in the **Input** section.

1. `number of external legs, external momenta, external indices`: Specify the number of external legs, and the momentum and corresponding index for each leg.
2. `number of internal indices, internal indices`: Specify the number of internal indices and list them.
3. `list of momenta, list of indices`: For each bare vertex, list the incoming momentum for each leg and the corresponding index.
4. `list of momenta, list of indices`: For each corrected 3-point vertex, corrected 4-point vertex and corrected 5-point vertex, list the incoming momentum for each leg and the corresponding index.
5. `list of momenta, list of indices`: List the momentum and the corresponding pair of initial and final indices for each propagator.
   • In this example, there is one corrected 3-point vertex and one corrected 4-point vertex. The corresponding momentum arguments are listed in double set brackets, indicating that the one vertex is the first in a list of length one. The same notation is used in any case where multiple entries would create nested lists. This includes the momenta and indices of bare or corrected n-point vertices, with $n \geq 3$, and the indices of propagators. The momenta for any number of propagators (even if there is only one) are listed in single set brackets.
6. `choose basis (onetwo, Keldysh or RA)`: Indicate the basis as either *onetwo*, *Keldysh* or *RA*.
7. `combination(s) to be evaluated: ie. ra, ... or AR, . or All`: Indicate the set of external indices of the diagrams that are to be calculated or simply specify *All*.
8. `Simplify result? (option to use the Mathematica function 'Simplify')`: The user can choose to have the program apply the *Mathematica* function *Simplify* to the result (*simplifyIt* = *yes*). For smaller diagrams this produces neater results. For larger diagrams it can lead to significantly longer running times, without producing a result that is much more compact.
9. `use the KMS conditions`: In the Keldysh or *RA* bases, one can choose to enforce the KMS conditions (*useKMSconditions=yes*). In the 1-2 basis choosing (*useKMSconditions=yes*) has no effect on the output. If the user has chosen the Keldysh basis, he can enforce the KMS conditions for the propagator, but not for higher n-point vertices (*removeFs=yes*). Note that if the user chooses *useKMSconditions=yes* and *removeFs=no*, the first choice over-rides the second, and KMS conditions are used for all n-point functions including propagators.
10. `express answer in terms of RA expressions`: In the Keldysh basis, if the calculation was done using the KMS conditions, the user can request that the right-hand side of the final expression be written in the *R/A* basis (*InTermsOfRA* = *yes*), since this basis frequently produces more compact results. The default value of this parameter is "no" and produces results in which both sides of the equation are written in the same basis.
11. `remove terms which are zero after integration`: One can choose to eliminate terms which will vanish after integration (*removeZeros=yes*). This option only works in the Keldysh or *RA* bases (in the 1-2 basis choosing (*removeZeros=yes*) has no effect on the output). The program identifies closed loops in the diagram, and then looks for terms in which all of the propagators that form a closed loop have poles on the same side of the real axis. Choosing (*removeZeros=yes*) will cause the program to set all of these terms (except for tadpoles) to zero. The program automatically exempts tadpole loops, so that they are not incorrectly set to zero. The list of combinations of propagators which will be removed by this option are stored in (*loopzerosubs*) and printed in the **Find Loops** section of the program. The propagators that correspond to tadpoles are also listed. One can compare these lists to the original diagram as a check of the information given in the **Input** section. For our example we have:

$$G_{\rm ra}(L)G_{\rm ra}(L+P+Q) \to 0\,,$$
$$G_{\rm ar}(L)G_{\rm ar}(L+P+Q) \to 0\,. \qquad (73)$$

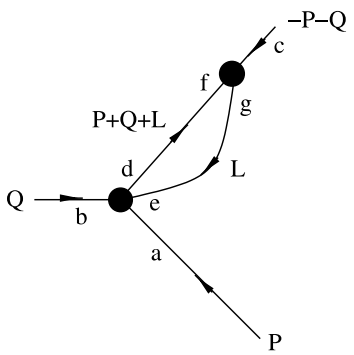

Fig. 3. Example Feynman diagram. *Uppercase letters* are momenta and *lowercase letters* label indices of the propagators and the vertices

Input

```
Off[General::spell];
Off[General::spell1];

(* number of external legs: *)      numExternalLegs = 3;
(* external momenta: *)             externalMomenta = {P, Q, -P - Q};
(* external indices: *)             externalIndices = {a, b, c};
(* number of internal indices: *)   numInternalIndices = 4;
(* internal indices: *)             internalIndices = {d, e, f, g};

(* list of momenta: *)              bare3ptMomenta = {};
(* list of indices: *)              bare3ptIndices = {};

(* list of momenta: *)              bare4ptMomenta = {};
(* list of indices: *)              bare4ptIndices = {};

  (* list of momenta: *)             Cor3ptMomenta = {{P + Q + L, -L, -P - Q}};
(* list of indices: *)              Cor3ptIndices = {{f, g, c}};

  (* list of momenta: *)             Cor4ptMomenta = {{P, Q, -P - Q - L, L}};
(* list of indices: *)              Cor4ptIndices = {{a, b, d, e}};

(* list of momenta: *)              Cor5ptMomenta = {};
(* list of indices: *)              Cor5ptIndices = {};

(* list of momenta: *)              propagatorMomenta = {L, P + Q + L};
(* list of indices: *)              propagatorIndices = {{g, e}, {d, f}};

(* choose basis (onetwo,
    Keldysh or RA) *)                                          Basis = Keldysh;
(* combination(s) to be evaluated: ie. {ra, ...} or {AR,.} or
    All *)                                                     combinations = All;
 (* Simplify result? (using Mathematica function:
      Simplify) *)                                           simplifyIt = no;

 (* use the KMS conditions *)
 useKMSconditions = no;
 (* use KMS conditions for propagator only
   (in the Keldysh basis) *)                                 removeFs = yes;
 (* express answer in terms of RA expressions
   (in equilibrium and Keldysh only) *)                      InTermsOfRA = no;

 (* remove terms which are zero after integration *)         removeZeros = yes;
 (* replace C functions in terms of N
   functions *)                                              replaceCs = yes;
(* specify name of file to output to using
   quotes *)
 FileName = "leafA_example.txt";

 (* Directory for output file is directory of this notebook as specified below *)
 SetDirectory[ DirectoryName[ ToFileName[NotebookInformation[InputNotebook[]][[1, 2]]] ] ];
```

Fig. 4. Input cell

The two expressions correspond to the one loop in our example diagram, with the momentum routed clockwise or counter-clockwise.

12. `replace C functions in terms of N functions`: If the KMS conditions are used, the program does the calculation in terms of the C-functions defined in (26). One can choose to replace these C-functions with their definitions in terms of thermal functions (*replaceCs=yes*) – see (26).
13. `specify name of output file using quotes`: The user can specify a filename to output the results to, or specify an empty string (“”) for no output file.
14. `Directory for output file is directory of this notebook as specified below`: The directory of the output file is set to be the same as the directory of the original notebook. The user can specify another directory for the output file by rewriting this line.

8.3 Error messages

Some checks have been implemented to detect possible input mistakes, and give error messages indicating the nature of the problem. A few examples are given below:

1. If the incoming momenta for any of the internal vertices, or for the external legs, do not sum to zero, the error message appears: "Momentum non-conserving vertex detected".
2. If the combinations of external indices requested using "`combination(s) to be evaluated`" do not correspond to the basis specified using "`choose basis (onetwo, Keldysh or RA)`", an error message appears. For example, choosing the RA basis and requesting the combination rra (corresponding to the vertex function Γ^{rra}) produces the error message: "Initialization Failed: Combination rra is not specified in the RA basis".
3. If the combinations of external indices requested do not have the same number of variables as the number of external legs, an error message will appear. For example, calculating a 3-point vertex function (as in our example) and requesting the combination $rraa$ (corresponding to the 4-point vertex Γ^{rraa}) produces the error message: "Initialization Failed: Combination $rraa$ does not have the correct number of indices".
4. Each external index must appear one time only as a vertex index. Each internal index must appear one time as a vertex index and one time as a propagator index. If the indices are incorrectly entered, an error message will appear.
5. Because of the fact that vertices are defined with incoming momenta, for each propagator, the momentum corresponding to the second index must be the same as the momentum for the vertex leg with the matching index. Similarly, the momentum corresponding to the first index must be the negative of the momentum for the vertex leg with the matching index. If the momentum variables are incorrectly entered, an error message will appear.

8.4 Interactive manipulation of the results

The result of the calculation is stored in the internal variable "*diagram*". The user can extract and manipulate the results in several ways.

1. One can extract a specific result using the function $DIAGRAM$. For our example, evaluating $DIAGRAM[a, r, r]$ gives the result for Γ^{arr} in the Keldysh representation:

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{i}G_{\mathrm{ar}}(L)G_{\mathrm{ra}}(L+P+Q)\Gamma^{\mathrm{arrr}}(P,Q,L,-L-P-Q)\\
&\times\Big[\Gamma^{\mathrm{aar}}(L+P+Q,-L,-P-Q)\\
&-N_L\Gamma^{\mathrm{arr}}(L+P+Q,-L,-P-Q)\\
&+N_{L+P+Q}\Gamma^{\mathrm{rar}}(L+P+Q,-L,-P-Q)\Big] \qquad (74)
\end{aligned}
$$

   (the prefactor $\int\frac{\mathrm{d}^4l}{(2\pi)^4}$ is not explicitly written).
2. In (79) we define notation for the Keldysh basis so that each string of indices of the form $rrar\cdots$ corresponds to a single numerical index. A function called GAMMA is defined that will automatically supply the result of the calculation for a given numerical index. All vertices are translated into the notation defined in (79). Propagators are written using the notation $D_{ra}(P)=D_{\mathrm{ret}}(P):=r(P)$, $D_{ar}(P)=D_{\mathrm{adv}}(P):=a(P)$. The function GAMMA is illustrated below. From (79), the index '2' corresponds to the string 'arr' for a 3-point vertex function. For our example, evaluating GAMMA[2] produces the translation of the result for Γ^{arr} that is given in (74):

$$
\begin{aligned}
&-\frac{1}{2}\mathrm{i}a(L)r(L+P+Q)\mathrm{M}(2,P,Q,L,-L-P-Q)\\
&\times\Big[\Gamma(4,L+P+Q,-L,-P-Q)\\
&-N_L\Gamma(2,L+P+Q,-L,-P-Q)\\
&+N_{L+P+Q}\Gamma(3,L+P+Q,-L,-P-Q)\Big] \qquad (75)
\end{aligned}
$$

9 The QED ward identity for the 4-point function

In this section, we present a calculation of the ward identity for the QED 3-point vertex function, and the QED 4-point vertex function involving two photons and two fermions. Throughout this section we work in the Keldysh representation and use Σ to denote the fermion self-energy. We work with an arbitrary effective theory (which could be, for example, the hard thermal loop theory) and use solid dots to indicate effective vertices.

As explained in the beginning of Sect. 8, the goal of our program is provide compact results for the expressions that result from summing over CTP indices, and the dirac structure of fermions and any group structure associated with all fields, must be handled separately by the user. For the calculations done in this paper, there are no additional complications. The ward identities are derived by comparing groups of integrands, without evaluating the integrals themselves. Consequently, we can simply suppress all Dirac and Lorentz indices. In addition, we multiply the result of the program by a factor $(-1)^{(\#\text{ photons})}$ where "# photons" is the number of photon lines in the diagram. The origin of this factor is as follows. When converting a scalar line to a photon line we use the usual convention shown in Fig. 5, which gives $\mathrm{i}D(p)\to-\mathrm{i}D_{\mu\nu}(p)$. Thus, for each photon line, in addition to including the appropriate projection operator, we must also include a factor of (-1).

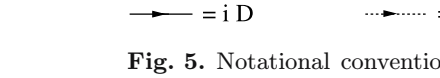

Fig. 5. Notational conventions for scalar and photon propagators

We illustrate these points with one example. The result for the contribution to $\Sigma(2,p)$ shown in Fig. 7 is given by (87). Note that the result produced by the program has been multiplied by a factor of (-1), because the diagram contains one photon line. The first term is $-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}a(l)r(l+p)\Gamma(2,p,l+p)\Gamma(4,l+p,p)$. Including

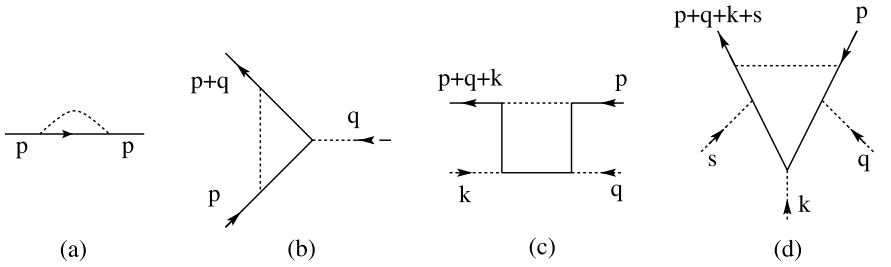

Fig. 6. One loop diagrams: *solid lines* are fermions and *dotted lines* are photons

all indices, this term is written:

$$-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}a_{\mu\nu}(l)\Gamma^{\mu}_{\beta\beta'}(2,p,l+p)r_{\beta'\alpha'}(l+p)\\ \Gamma^{\nu}_{\alpha'\alpha}(4,l+p,p)\,, \tag{76}$$

where $\{\mu,\nu\}$ are Lorentz indices and $\{\alpha,\alpha',\beta,\beta'\}$ are Dirac indices.

Throughout the rest of this section we will leave all integrands in the form described above, with dirac and lorentz indices suppressed. In addition, we define

$$k+p+q=u\,,\quad k+p=t\,,\quad p+q=s\,, \tag{77}$$

and use the shorthand notation: $D_{\mathrm{ret}}(p)=r(p)$, $D_{\mathrm{adv}}(p)=a(p)$, $D_{\mathrm{sym}}(p)=f(p)$. We do not consider tadpole diagrams, because their structure is essentially uninteresting. The basic reason is that the introduction of a tadpole does not change the pole structure of the existing propagators. This point is explained in detail for one example in Appendix.

In order to simplify the notation for the vertices, we replace each combination of the indices $\{r,a\}$ by a single numerical index:

$$\Gamma^{(n)\,\alpha_1\alpha_2\cdots\alpha_n}(p_1,p_2,\cdots p_n)=\Gamma^{(n)}(i,p_1,p_2,\cdots p_n) \tag{78}$$

We assign the choices of the variables $\alpha_1\alpha_2\cdots\alpha_n$ to the variable i using the vector

$$V_n=\begin{pmatrix}r_n\\a_n\end{pmatrix}\cdots\otimes\begin{pmatrix}r_2\\a_2\end{pmatrix}\otimes\begin{pmatrix}r_1\\a_1\end{pmatrix}, \tag{79}$$

where the symbol $\otimes$ indicates the outer product. For each n, the ith component of the vector corresponds to a list of variables that is assigned the number i. To simplify the notation we drop the subscripts and write a list like $r_1r_2a_3$ as rra. For clarity, the results are listed below.

[a] 2-point functions: $rr\to1$, $ar\to2$, $ra\to3$, $aa\to4$

[b] 3-point functions: $rrr\to1$, $arr\to2$, $rar\to3$, $aar\to4$, $rra\to5$, $ara\to6$, $raa\to7$, $aaa\to8$

[c] 4-point functions: $rrrr\to1$, $arrr\to2$, $rarr\to3$, $aarr\to4$, $rrar\to5$, $arar\to6$, $raar\to7$, $aaar\to8$, $rrra\to9$, $arra\to10$, $rara\to11$, $aara\to12$, $rraa\to13$, $araa\to14$, $raaa\to15$, $aaaa\to16$

[d] 5-point functions: $rrrrr\to1$, $arrrr\to2$, $rarrr\to3$, $aarrr\to4$, $rrarr\to5$, $ararr\to6$, $raarr\to7$, $aaarr\to8$, $rrrar\to9$, $arrar\to10$, $rarar\to11$, $aarar\to12$, $rraar\to13$, $araar\to14$, $raaar\to15$, $aaaar\to16$, $rrrra\to17$, $arrra\to18$, $rarra\to19$, $aarra\to20$, $rrara\to21$, $arara\to22$, $raara\to23$, $aaara\to24$, $rraa\to25$, $arraa\to26$, $raraa\to27$, $aaraa\to28$, $rraaa\to29$, $araaa\to30$, $raaaa\to31$, $aaaaa\to32$

We note that $i=1$ corresponds to a vertex that is identically zero, for any number of external legs, as a consequence of the general relation $\Gamma^{rrr\cdots r}=0$ which can be obtained from (66).

To reduce the number of indices we will introduce separate names for the first five vertex functions and write:

$$\Gamma^{(2)}=\Sigma\,,\quad \Gamma^{(3)}=\Gamma\,,\quad \Gamma^{(4)}=M\,,\quad \Gamma^{(5)}=C\,. \tag{80}$$

The notation for the momentum arguments is as follows:

$$\Sigma(p_{\mathrm{in}})\,,\quad \Gamma^{\mu}(p_{\mathrm{in}},p_{\mathrm{out}})\,,\quad M^{\mu\nu}(p_{\mathrm{in}},q_1^{\mu},q_2^{\nu},p_{\mathrm{out}})\,,\\ C^{\mu\nu\tau}(p_{\mathrm{in}},q_1^{\mu},q_2^{\nu},q_3^{\tau},p_{\mathrm{out}})\,, \tag{81}$$

where p_{in} is the momentum of the incoming fermion, $\{q_1,q_2,q_3\}$ are the momenta of the incoming photons, and p_{out} is the momentum of the outgoing fermion. Note that the momentum of the photon is not written for the 3-point vertex since it can be inferred from the momenta of the fermions. Similarly, the self-energy is written $\Sigma(p_{\mathrm{in}})$ instead of $\Sigma(p_{\mathrm{in}},-p_{\mathrm{in}})$.

We begin by calculating the ward identity at the bare 1-loop level. In the next section, we verify that the same ward identities are satisfied by the complete set of graphs involving full corrected vertices.

9.1 Bare 1-loop diagrams

We start by looking at 1-loop diagrams with bare vertices, in order to determine the form of the ward identities. For the 2- and 3-point functions the graphs are shown in Fig. 6a and b. For the 4-point function we have the box graph (Fig. 6c), and the crossed version of the box graph where the two external photons are interchanged. For the five point function, the basic graph is show in Fig. 6d. There are six versions of this graph which correspond to the six possible permutations of the three external photons.

We use the notation defined in Fig. 2 and insert the appropriate coupling constants for QED.

$$\begin{aligned}
&-\mathrm{i}\Sigma^{(1)} = \text{self-energy graph}\\
&-\mathrm{ie}\Gamma^{(1)} = (\text{triangle graph})\\
&-\mathrm{ie}^2 M^{(1)} = (\text{box graph} + \text{crossed box graph})\\
&-\mathrm{ie}^3 C^{(1)} = (\text{six permutations of 5-point graph})
\end{aligned} \tag{82}$$

where the superscripts refer to the loop order of the graph. The ward identities are of the form

$$\begin{aligned}
&Q^\mu \Gamma^{(1)}_\mu(i,p,p+q)\\
&= \Sigma^{(1)}(j_1,p) - \Sigma^{(1)}(j_2,p+q)\\
&Q^\mu M^{(1)}_{\mu\nu}(i,p,q,k,p+q+k)\\
&= \Gamma^{(1)}_\nu(j_1,p,p+k) - \Gamma^{(1)}_\nu(j_2,p+q,k+p+q)\\
&Q^\mu C^{(1)}_{\mu\nu\tau}(i,p,q,k,s,p+q+k+s)\\
&= M^{(1)}_{\nu\tau}(j_1,p,k,s,k+p+s)\\
&\quad - M^{(1)}_{\nu\tau}(j_2,p+q,k,s,k+p+q+s)
\end{aligned} \tag{83}$$

where the indices $\{i, j_1, j_2\}$ refer to the choices of the variables $\alpha_1\alpha_2\cdots\alpha_n$ as defined in (79). We list below the sets of these indices for the 3-point, 4-point and 5-point functions.

$$\begin{aligned}
\Gamma :&(2,2,2)\,,(3,3,2)\,,(4,4,1)\,,(5,3,3)\,,(6,4,4)\,,(7,1,4)\,,\\
&(8,2,3)\\
M :&(2,2,2)\,,(3,5,2)\,,(4,6,1)\,,(5,3,3)\,,(6,4,4)\,,(7,7,4)\,,\\
&(8,8,3)\,,(9,5,5)\,,(10,6,6)\,,(11,1,6),\\
&(12,2,5)\,,(13,7,7)\,,(14,8,8)\,,(15,3,8)\,,(16,4,7)\\
C :&(2,2,2)\,,(3,9,2)\,,(4,10,1)\,,(5,3,3)\,,(6,4,4)\,,(7,11,4)\,,\\
&(8,12,3)\,,(9,5,5)\,,(10,6,6)\,,(11,13,6),\\
&(12,14,5)\,,(13,7,7)\,,(14,8,8)\,,(15,15,8)\,,(16,16,7)\,,\\
&(17,9,9)\,,(18,10,10)\,,(19,1,10)\,,(20,2,9),\\
&(21,11,11)\,,(22,12,12)\,,(23,3,12)\,,(24,4,11)\,,\\
&(25,13,13)\,,(26,14,14)\,,(27,5,14)\,,(28,6,13)\,,\\
&(29,15,15)\,,(30,16,16)\,,(31,7,16)\,,(32,8,15)\,.
\end{aligned} \tag{84}$$

We give three specific examples below. The first set of numbers in the first line of (84) is (2,2,2). Using (79) and (83) the corresponding ward identity is:

$$\begin{aligned}
&Q^\mu \Gamma^{(1)}_\mu(2,p,p+q) = \Sigma^{(1)}(2,p) - \Sigma^{(1)}(2,p+q)\\
\text{or}\quad &Q\cdot\left(\Gamma^{(1)}\right)^{arr}(p,p+q) = \left(\Sigma^{(1)}\right)^{ar}(p)\\
&\qquad - \left(\Sigma^{(1)}\right)^{ar}(p+q)
\end{aligned} \tag{85}$$

The first set of numbers in the second line of (84) is (2,2,2). The corresponding ward identity is

$$\begin{aligned}
Q^\mu M^{(1)}_{\mu\nu\tau}(2,p,q,k,p+q+k) &= \Gamma^{(1)}_{\nu\tau}(2,p,p+k)\\
&\quad - \Gamma^{(1)}_{\nu\tau}(2,p+q,k,k+p+q)
\end{aligned}$$

or

$$\begin{aligned}
Q^\mu\left(M^{(1)}_{\mu\nu\tau}\right)^{arrr}(p,p+k) &= \left(\Gamma^{(1)}_{\nu\tau}\right)^{arr}(p,k+p)\\
&\quad - \left(\Gamma^{(1)}_{\nu\tau}\right)^{arr}(p+q,k+p+q)
\end{aligned} \tag{86}$$

The last set of numbers in the last line of (84) is (32,8,15). The corresponding ward identity is

$$\begin{aligned}
&Q^\mu C^{(1)}_{\mu\nu\tau}(32,p,q,k,s,p+q+k+s)\\
&= M^{(1)}_{\nu\tau}(8,p,k,s,k+p+s)\\
&\quad - M^{(1)}_{\nu\tau}(15,p+q,k,s,k+p+q+s)
\end{aligned}$$

or

$$\begin{aligned}
&Q^\mu\left(C^{(1)}_{\mu\nu\tau}\right)^{aaaaa}(p,q,k,s,p+q+k+s)\\
&= \left(M^{(1)}_{\nu\tau}\right)^{aaar}(p,k,s,k+p+s)\\
&\quad - \left(M^{(1)}_{\nu\tau}\right)^{raaa}(p+q,k,s,k+p+q+s)\,.
\end{aligned}$$

9.2 Full vertices

We verify that the ward identities derived above hold for the full 3-point and 4-point vertex functions.

9.2.1 2-point vertex function

We give the results for the 2-point vertex function shown in Fig. 7. These expressions will be needed to verify the ward identities for the 3-point vertex function.

$$\begin{aligned}
\Sigma(2,p) = -\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,&a(l)r(l+p)\Gamma(2,p,l+p)\\
&\times(\Gamma(4,l+p,p)+\Gamma(3,l+p,p)N_F(l+p)\\
&-\Gamma(2,l+p,p)N_B(l))\,]\\
\Sigma(3,p) = -\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,&a(l+p)r(l)\Gamma(5,l+p,p)\\
&\times(\Gamma(7,p,l+p)-\Gamma(3,p,l+p)N_F(l+p)\\
&+\Gamma(5,p,l+p)N_B(l))\,]\\
\Sigma(4,p) = -\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,&a(l+p)r(l)\Gamma(5,l+p,p)\\
&\times(\Gamma(8,p,l+p)-\Gamma(4,p,l+p)N_F(l+p)\\
&+(\Gamma(6,p,l+p)-\Gamma(2,p,l+p)N_F(l+p))\\
&\times N_B(l))\\
&+a(l)r(l+p)\Gamma(2,p,l+p)(\Gamma(8,l+p,p)\\
&+\Gamma(7,l+p,p)N_F(l+p)-(\Gamma(6,l+p,p)\\
&+\Gamma(5,l+p,p)N_F(l+p))N_B(l))\,]
\end{aligned} \tag{87}$$

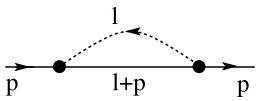

Fig. 7. The 2-point vertex function

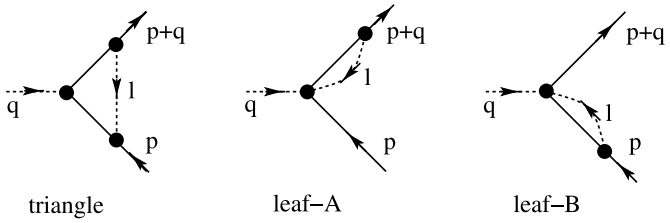

Fig. 8. The 3-point vertex function

9.3 3-point vertex function

First we calculate all seven components of the three graphs shown in Fig. 8 that contribute to the 3-point vertex function. We give the result for one example: $\Gamma(2,p,p+q)$.

$$
\begin{aligned}
&\Gamma_{\text{triangle}}(2,p,p+q)\\
&=-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p)\\
&\qquad\times(a(l+s)\Gamma(3,l+s,s)(\Gamma(6,l+p,l+s)\\
&\qquad+\Gamma(5,l+p,l+s)N_F(l+p)\\
&\qquad-\Gamma(2,l+p,l+s)N_F(l+s))\\
&\qquad+r(l+s)\Gamma(2,l+p,l+s)(\Gamma(4,l+s,s)\\
&\qquad+\Gamma(3,l+s,s)N_F(l+s)-\Gamma(2,l+s,s)N_B(l)))]\\
&\Gamma_{\text{leafA}}(2,p,p+q)\\
&=-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,[\,a(l)r(l+s)M(2,p,q,l,l+s)(\Gamma(4,l+s,s)\\
&\qquad+\Gamma(3,l+s,s)N_F(l+s)-\Gamma(2,l+s,s)N_B(l))\,]\\
&\Gamma_{\text{leafB}}(2,p,p+q)\\
&=-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p)\\
&\qquad\times(M(6,l+p,q,-l,s)+M(5,l+p,q,-l,s)\\
&\qquad\times N_F(l+p)-M(2,l+p,q,-l,s)N_B(l))\,]\,. \qquad (88)
\end{aligned}
$$

Next, we verify the ward identities for the seven vertex functions. We give detailed results for one example: $\Gamma(2,p,p+q)$. Starting from (88) and contracting with Q we obtain:

$$
\begin{aligned}
Q\cdot\Gamma_{\text{triangle}}(2,p,p+q)&=\mathcal{X}[2]+\mathcal{Y}[2]\\
Q\cdot\Gamma_{\text{leafA}}(2,p,p+q)&=-\mathcal{X}[2]+\mathbf{x}[2]\\
Q\cdot\Gamma_{\text{leafB}}(2,p,p+q)&=-\mathcal{Y}[2]+\mathbf{y}[2]\,, \qquad (89)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{X}[2]&=\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}[a(l)r(l+s)\Gamma(2,p,l+p)(\Gamma(4,l+s,s)\\
&\qquad+\Gamma(3,l+s,s)N_F(l+s)\\
&\qquad-\Gamma(2,l+s,s)N_B(l))]\\
\mathcal{Y}[2]&=-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p)(\Gamma(4,l+s,s)\\
&\qquad+\Gamma(3,l+s,s)N_F(l+p)\\
&\qquad-\Gamma(2,l+s,s)N_B(l))]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{x}[2]&=\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,[\,a(l)r(l+s)\Gamma(2,s,l+s)\\
&\qquad\times(\Gamma(4,l+s,s)+\Gamma(3,l+s,s)N_F(l+s)\\
&\qquad-\Gamma(2,l+s,s)N_B(l))\,]\\
\mathbf{y}[2]&=-\frac{\mathrm{i}}{2}\int\frac{\mathrm{d}^4l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p)(\Gamma(4,l+p,p)\\
&\qquad+\Gamma(3,l+p,p)N_F(l+p)\\
&\qquad-\Gamma(2,l+p,p)N_B(l))]\,. \qquad (90)
\end{aligned}
$$

Comparing with (87) we obtain

$$
\begin{aligned}
Q\cdot\Gamma(2,p,p+q)&=Q\big(\Gamma_{\text{triangle}}(2,p,p+q)\\
&\qquad+\Gamma_{\text{leafA}}(2,p,p+q)+\Gamma_{\text{leafB}}(2,p,p+q)\big)\\
&=\Sigma(2,p)-\Sigma(2,p+q)\,,
\end{aligned}
$$

which agrees with (85). The results for all components agree with the results listed in (84).

9.3.1 4-point vertex function

We verify the ward identity for the 15 4-point vertex functions. We give detailed results for one example: $M(2,p,q,k,p+q+k)$. There are five types of diagrams to consider. They are shown in Figs. 9–13.

Contracting with Q and using (89) and (90) we obtain:

$$
\begin{aligned}
\Gamma_{\text{seagull-1}}(2,p,q,k,u)&=\alpha[2]+A[2]\,,\\
\Gamma_{\text{seagull-2}}(2,p,q,k,u)&=\beta[2]+B[2]\,,\\
\Gamma_{\text{leaftail-1}}(2,p,q,k,u)&=\gamma[2]+C[2]\,,\\
\Gamma_{\text{leaftail-3}}(2,p,q,k,u)&=\delta[2]+D[2]\,,\\
\Gamma_{\text{jellyfish-1}}(2,p,q,k,u)&=\epsilon[2]+E[2]\,,\\
\Gamma_{\text{jellyfish-2}}(2,p,q,k,u)&=\phi[2]+F[2]\,,\\
\Gamma_{\text{polywog}}(2,p,q,k,u)&=G[2]+H[2]\,,\\
\Gamma_{\text{leaftail-2}}(2,p,q,k,u)&=-B[2]-F[2]\,,\\
\Gamma_{\text{leaftail-4}}(2,p,q,k,u)&=-A[2]-E[2]\,,\\
\Gamma_{\text{box}}(2,p,q,k,u)&=-G[2]-C[2]\,,\\
\Gamma_{\text{crossed-box}}(2,p,q,k,u)&=-H[2]-D[2]\,, \qquad (91)
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha[2]&=-\Gamma_{\text{leafB}}(2,p+q,u)\\
\beta[2]&=\Gamma_{\text{leafA}}(2,p,t)\\
\gamma[2]&=-\Gamma_{\text{triangle}}(2,p+q,u)\\
\delta[2]&=\Gamma_{\text{triangle}}(2,p,t)\\
\epsilon[2]&=\Gamma_{\text{leafB}}(2,p,t)\\
\phi[2]&=-\Gamma_{\text{leafA}}(2,p+q,u)
\end{aligned}
$$

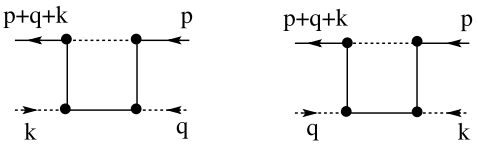

Fig. 9. The box and crossed-box diagrams

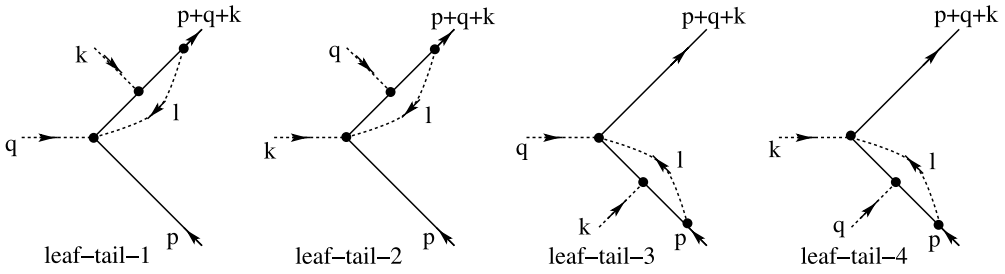

Fig. 10. The leaftail diagrams

$$A[2] = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+s)\Gamma(2,p,l+p) \times (M(6,l+s,k,-l,u)+M(5,l+s,k,-l,u) \times N_F(l+s) - M(2,l+s,k,-l,u)N_B(l))\,]$$

$$B[2] = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+t)M(2,p,k,l,l+t) \times (\Gamma(4,l+u,u)+\Gamma(3,l+u,u)N_F(l+t) - \Gamma(2,l+u,u)N_B(l))]$$

$$C[2] = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+s)\Gamma(2,p,l+p) \times (a(l+u)\Gamma(3,l+u,u)(\Gamma(6,l+s,l+u) + \Gamma(5,l+s,l+u)N_F(l+s) - \Gamma(2,l+s,l+u)N_F(l+u)) + r(l+u)\Gamma(2,l+s,l+u)(\Gamma(4,l+u,u) + \Gamma(3,l+u,u)N_F(l+u) - \Gamma(2,l+u,u) \times N_B(l)))]$$

$$D[2] = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p) \times (a(l+t)\Gamma(3,l+u,u)(\Gamma(6,l+p,l+t) + \Gamma(5,l+p,l+t)N_F(l+p) - \Gamma(2,l+p,l+t) \times N_F(l+t)) + r(l+t)\Gamma(2,l+p,l+t)(\Gamma(4,l+u,u) + \Gamma(3,l+u,u)N_F(l+t) - \Gamma(2,l+u,u)N_B(l)))]$$

$$E[2] = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}[a(l)r(l+p)\Gamma(2,p,l+p) \times (M(6,l+s,k,-l,u)+M(5,l+s,k,-l,u) \times N_F(l+p) - M(2,l+s,k,-l,u)N_B(l))]$$

$$F[2] = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+u)M(2,p,k,l,l+t) \times (\Gamma(4,l+u,u)+\Gamma(3,l+u,u)N_F(l+u) - \Gamma(2,l+u,u)N_B(l))]$$

$$G[2] = \frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p) \times (a(l+u)\Gamma(3,l+u,u)(\Gamma(6,l+s,l+u) + \Gamma(5,l+s,l+u)N_F(l+p) - \Gamma(2,l+s,l+u)N_F(l+u)) + r(l+u)\Gamma(2,l+s,l+u)(\Gamma(4,l+u,u) + \Gamma(3,l+u,u)N_F(l+u) - \Gamma(2,l+u,u)N_B(l)))]$$

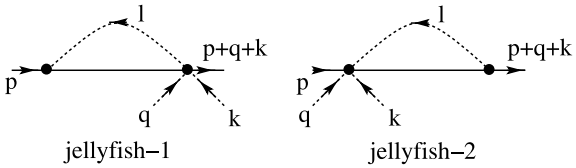

Fig. 11. The jellyfish diagrams

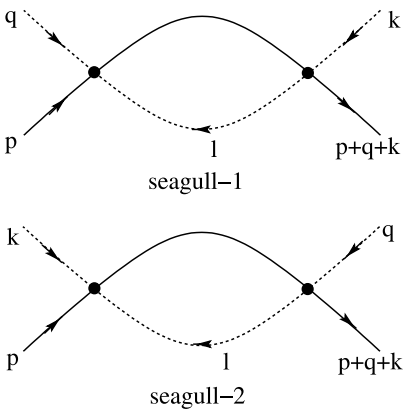

Fig. 12. The seagull diagrams

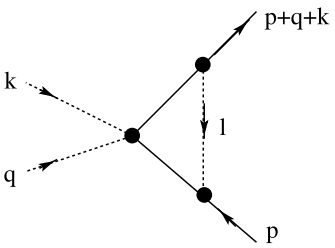

Fig. 13. The polywog diagram

$$H[2] = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\,[\,a(l)r(l+p)\Gamma(2,p,l+p) \times (a(l+u)\Gamma(3,l+u,u)(\Gamma(6,l+p,l+t) + \Gamma(5,l+p,l+t)N_F(l+p) - \Gamma(2,l+p,l+t)N_F(l+u)) + r(l+u)\Gamma(2,l+p,l+t)(\Gamma(4,l+u,u) + \Gamma(3,l+u,u)N_F(l+u) - \Gamma(2,l+u,u)N_B(l)))]\,. \quad (92)$$

Using $\Gamma = \Gamma_{\mathrm{triangle}} + \Gamma_{\mathrm{leafA}} + \Gamma_{\mathrm{leafB}}$ and combining we have

$$Q^\mu M_{\mu\nu}(2,p,q,k,u) = \Gamma_\nu(2,p,t) - \Gamma_\nu(2,p+q,u)\,.$$

which agrees with (86). The results for all components agree with the results listed in (84).

10 Conclusions

Calculations in real time statistical field theory are complicated by the extra indices that result from the doubling of degrees of freedom. Because of this technical problem, many people avoid the real time formalism of finite temperature field theory in spite of its significant advantages, as compared with the imaginary time formalism. Two of the major advantages of working in real time are the fact that analytic continuations are not necessary, and that it is easy to generalize to non-equilibrium situations. In this paper we have made a contribution towards reducing the technical difficulties associated with the real time formulation of statistical field theory. We have written a *Mathematica* program that performs the contractions over the tensor indices that appear in real time statistical field theory and determines the integrand corresponding to any amplitude. The program is designed so that it can be used by someone with no previous experience with *Mathematica*. It is available on the internet at www.brandonu.ca/physics/fugleberg/Research/Dick.html. It can be used in the 1-2, Keldysh or *RA* basis, and it can do calculations in or out of equilibrium.

We have used the program to calculate the QED ward identity for the 3-point function ($2^n - 1 = 7$ components), the 4-point function for two fermions and two photons (($2^n - 1 = 15$ components), and the 5-point function for two fermions and three photons ($2^n - 1 = 31$ components). Some of these identities have appeared previously in the literature, but the complete set of identities has not previously been published. The calculation therefore serves two purposes: it provides a check of the program, and it produces useful new information. We give a table that lists the results for the ward identities (84). In addition, we give a simple general expression for the KMS conditions between n-point functions and vertex functions, in both the Keldysh and *RA* bases ((60) and (67)).

Appendix : Tadpoles

Our program can handle tadpole diagrams, but we do not explicitly consider them in the examples of the QED ward identity that we present in Sect. 9, since their structure is essentially uninteresting. We illustrate this point with an example. The ward identity for the 3-point function involves the 1-loop tadpole diagram shown in Fig. 14a.

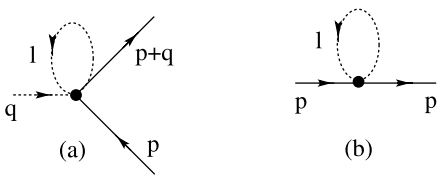

Fig. 14. Two tadpole diagrams

We give below the result for one component of this amplitude. Using the same notation as in Sect. 9, we suppress all dirac and lorentz indices and write the result:

$$\Gamma(5,p,p+q) = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\, f(l) C(17,p,q,l,-l,p+q)\,. \tag{A.1}$$

Contracting with Q and using (83) we obtain:

$$Q\cdot\Gamma(5,p,p+q) = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4} f(l)(M(9,p,l,-l,p) - M(9,p+q,l,-l,p+q)). \tag{A.2}$$

The tadpole graph shown in Fig. 14b gives:

$$\Sigma(3,p) = -\frac{\mathrm{i}}{2}\int \frac{\mathrm{d}^4 l}{(2\pi)^4}\, f(l) M(9,p,l,-l,p)\,. \tag{A.3}$$

Comparing (A.2) and (A.3) we have:

$$Q\cdot\Gamma(5,p,p+q) = \Sigma(3,p) - \Sigma(3,p+q)\,, \tag{A.4}$$

in agreement with the result given in (83). As noted above, the structure of these tadpole diagrams is essentially uninteresting, and therefore we have not included them in Sect. 9.

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